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THE DISTANCE BETWEEN HOMOTOPY CLASSES OF S^1-VALUED MAPS IN MULTIPLY CONNECTED DOMAINS

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ABSTRACT

Certain Sobolev spaces of S^1-valued functions can be written as a disjoint union of homotopy classes. The problem of finding the distance between different homotopy classes in such spaces is considered. In particular, several types of one-dimensional and two-dimensional domains are studied. Lower bounds are derived for these distances. Furthermore, in many cases it is shown that the lower bounds are sharp but are not achieved.

1. Introduction

Let D be a multiply connected domain or an embedded compact multiply connected manifold in $\mathbb{R}^N$. Suppose that the space $H^1(D,S^1)$ can be written as a disjoint union of homotopy classes

$$H^1(D,S^1)=\bigcup_{\boldsymbol{d}}\mathcal{E}_{\boldsymbol{d}}, \tag{1.1}$$

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so that the homotopy classes are indexed by vectors of integers $\boldsymbol{d}$. Such partitions of Sobolev spaces was first observed by White [7]. In particular, partitions like (1.1) indeed exist when D is the circle S^1, a planar graph, a compact multiply connected two-dimensional domain, etc. This partition found interesting applications in physics, where it was used in [6], to explain persistent currents in superconductivity, and to predict in [5], new structures in liquid crystals.

For $\boldsymbol{d}^{(1)} \neq \boldsymbol{d}^{(2)}$ set

$$\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) = \inf\left\{ \int_D |\nabla(u_1 - u_2)|^2 : u_1 \in \mathcal{E}_{\boldsymbol{d}^{(1)}}, u_2 \in \mathcal{E}_{\boldsymbol{d}^{(2)}} \right\}. \tag{1.1}$$

Two natural questions arise concerning this distance function:

(i) What is the value of $\delta(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)})$?

(ii) Is the infimum in (1.2) achieved?

In the next section we solve both questions for the case $D = S^1$. In section 3 we consider two-dimensional multiply connected domains D. We derive a general lower bound for $\delta(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)})$, and prove that the bound is sharp under certain conditions (called property (C)) on the homotopy class vectors. Finally, in section 4 we demonstrate through an example that the general lower bound derived in section 3 may not be optimal if condition (C) is not satisfied. Although some of our results can be readily extended to some three-dimensional domains (such as the solid torus), we have not examined in detail more general three-dimensional multiply connected domains.

2. Maps from S^1 to S^1

Note that $H^1(S^1, S^1) \subset C^{1/2}(S^1, S^1)$, so that each $u \in H^1(S^1, S^1)$ has a well defined degree, and we may write

$$H^1(S^1, S^1) = \bigcup_{d \in \mathbb{Z}} \mathcal{E}_d = \bigcup_{d \in \mathbb{Z}} \{u : \deg u = d\}.$$

The formula (1.2) for the distance between $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ reads in the current case,

$$\delta^2(d_1, d_2) = \inf\left\{ \int_{S^1} |(u_1 - u_2)'|^2 : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2} \right\}. \tag{2.1}$$

A simple lower-bound for $\delta(d_1, d_2)$ is given by the following lemma.

Lemma 2.1: *We have:*

$$\delta^2(d_1, d_2) \geq \frac{8(d_2 - d_1)^2}{\pi}.$$

Proof: Clearly it suffices to consider the case $m := d_2 - d_1 > 0$. For each pair $u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}$ we have $v = u_2/u_1 \in \mathcal{E}_m$; hence v covers S^1 (algebraically) m times. In particular, each of the values ± 1 is attained at least m times. It follows that there are points:

$$0 \le s_1 < t_1 < s_2 < t_2 < s_3 < \cdots < s_m < t_m < s_{m+1} = s_1 + 2\pi,$$

such that $v(e^{is_j}) = 1$ (i.e., $|u_2 - u_1|(e^{is_j}) = 0$) and $v(e^{it_j}) = -1$ (i.e., $|u_2 - u_1|(e^{it_j}) = 2$) for each j. A simple consideration and direct calculation gives

$$\int_{S^1} |(u_2 - u_1)'|^2 \ge \int_{S^1} \big||u_2 - u_1|'\big|^2 \ge \sum_{j=1}^{m} \Big(\frac{4}{t_j - s_j} + \frac{4}{s_{j+1} - t_j}\Big). \tag{2.2}$$

It is clear, for example from the arithmetic-harmonic means inequality, that the smallest value of the last expression is achieved when all the points $\{s_j, t_i\}$ are equally spaced. Therefore,

$$\int_{S^1} |(u_2 - u_1)'|^2 \ge \int_{S^1} \big||u_2 - u_1|'\big|^2 \ge \frac{(4m)^2}{2\pi} = \frac{8m^2}{\pi}. \tag{2.3}$$ ∎

The next simple lemma shows that the distance between two homotopy classes depends on the difference of the degrees only.

LEMMA 2.2: *We have* $\delta(d_1 + k, d_2 + k) = \delta(d_1, d_2)$, $\forall k \in \mathbb{Z}$.

Proof: Take any $u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}$ with $m = d_2 - d_1 \ne 0$. With a slight abuse of notation we view each u_j also as a map from $[0, 2\pi]$ to S^1 satisfying $u_j(0) = u_j(2\pi)$. Since the image of u_2/u_1 is the whole circle S^1, the point 1 is in this image, and so we may assume without loss of generality that $u_1(0) = u_2(0)$. For a small $\varepsilon > 0$ define the "rescaled" maps $\tilde{u}_j = \tilde{u}_j^{(\varepsilon)}$ on $[0, 2\pi - \varepsilon]$ by

$$\tilde{u}_j(\theta) = u_j\Big(\frac{2\pi}{2\pi - \varepsilon}\theta\Big), \quad \text{where } j = 1, 2. \tag{2.4}$$

On the remaining interval $[2\pi - \varepsilon, 2\pi]$ complete the definition of $\tilde{u}_1, \tilde{u}_2$ by

$$\tilde{u}_1(\theta) = \tilde{u}_2(\theta) = u_1(0) \cdot \exp\Big(2\pi k i \frac{\theta - (2\pi - \varepsilon)}{\varepsilon}\Big).$$

Clearly, $\tilde{u}_j \in \mathcal{E}_{d_j + k}, j = 1, 2$, and

$$\lim_{\varepsilon \to 0} \int_{S^1} |(\tilde{u}_2 - \tilde{u}_1)'|^2 = \int_{S^1} |(u_2 - u_1)'|^2.$$

The result follows since u_j can be chosen arbitrarily in $\mathcal{E}_{d_j}$. ∎

Next we give the main result of this section.

THEOREM 1: *For every $d_1, d_2 \in \mathbb{Z}$ we have:*
(i) $\delta^2(d_1,d_2) = \frac{8(d_2-d_1)^2}{\pi}$.
(ii) *For $d_1 \neq d_2$, $\delta(d_1,d_2)$ is not attained.*

Proof: (i) In view of Lemma 2.2 it suffices to consider two cases:
(1) $d_2 = d > 0, d_1 = -d$,
(2) $d_2 = d > 0, d_1 = -d+1$.
In each of these cases we put $m = d_2 - d_1$. Note that the equality

$$\int_{S^1} \left||u_2 - u_1|'\right|^2 = \frac{8m^2}{\pi} \tag{2.5}$$

is achieved in (2.2) if and only if the $2m+1$ points

$$s_1, t_1, s_2, t_2, \ldots, s_m, t_m, s_{m+1}$$

are equidistant and the graph of the function $|u_2 - u_1|$ is piecewise linear with vertices at the points $\{(s_j,0),(t_j,2)\}_{j=1}^m$. Motivated by the above, we set, for $j = 1,\ldots,m+1$,

$$\tilde{s}_j = (j-1)\frac{2\pi}{m} \quad \text{and} \quad \tilde{t}_j = (j-1)\frac{2\pi}{m} + \frac{\pi}{m},$$

and define the function ρ by

$$\rho(\theta) = \begin{cases} (\theta - \tilde{s}_j)\frac{2m}{\pi} & \theta \in [\tilde{s}_j, \tilde{t}_j], \\ 2 - (\theta - \tilde{t}_j)\frac{2m}{\pi} & \theta \in (\tilde{t}_j, \tilde{s}_{j+1}], \end{cases} \tag{2.6}$$

for $j = 1,\ldots,m$. For any small $\varepsilon > 0$ consider the following approximation ρ_ε of ρ:

$$\rho^{(\varepsilon)}(\theta) = 2J_\varepsilon\left(\frac{\rho}{2}\right),$$

where the map $J_\varepsilon\colon [-1,1] \to [-1,1]$ is an odd C^2-map enjoying the following properties:

$$\begin{array}{ll} J_\varepsilon(\pm 1) = \pm 1, & J'_\varepsilon(\pm 1) = 0, \\ J_\varepsilon(t) = t, & |t| \le 1-\varepsilon, \\ 0 < J'_\varepsilon(t) < c_0, & |t| < 1, \\ \frac{c_1}{\varepsilon} \le |J''_\varepsilon(t)| \le \frac{c_2}{\varepsilon}, & 1 - \frac{\varepsilon}{2} \le |t| \le 1, \end{array} \tag{2.7}$$

for some positive constants c_0, c_1, c_2 (independent of ε). Set $u_2^{(\varepsilon)}(\theta) = e^{i\alpha(\theta)}$ and then $u_1^{(\varepsilon)}(\theta) = \bar{u}_2^{(\varepsilon)}(\theta) = e^{-i\alpha(\theta)}$ where

$$\alpha(\theta) = \alpha^{(\varepsilon)}(\theta) = \begin{cases} \sin^{-1}(\rho^{(\varepsilon)}/2) - (1+(-1)^j)\frac{\pi}{2} & \theta \in [\tilde{s}_j, \tilde{t}_j], \\ \pi - \sin^{-1}(\rho^{(\varepsilon)}/2) - (1+(-1)^j)\frac{\pi}{2} & \theta \in (\tilde{t}_j, \tilde{s}_{j+1}], \end{cases} \tag{2.8}$$

for $j = 1, \dots, m$. Thanks to (2.7) we have $e^{i\alpha} \in Lip[0, 2\pi]$.

Consider first case (1). Then, $u_2^{(\varepsilon)} \in \mathcal{E}_d, u_1^{(\varepsilon)} \in \mathcal{E}_{-d}$ and

$$(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})(\theta) = \pm i|(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})(\theta)| = \pm i\rho^{(\varepsilon)}(\theta), \quad \text{for all } \theta. \tag{2.9}$$

Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{S^1} |(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})'|^2 &= \lim_{\varepsilon\to 0}\int_{S^1} ||u_2^{(\varepsilon)} - u_1^{(\varepsilon)}|'|^2 \\ &= \lim_{\varepsilon\to 0}\int_{S^1} |(\rho^{(\varepsilon)})'|^2 = \int_{S^1} (\rho')^2 = \frac{8(2d)^2}{\pi}.\end{aligned}$$

In case (2), the maps $u_1^{(\varepsilon)}, u_2^{(\varepsilon)}$ are well-defined as maps from $[0, 2\pi]$ to S^1, but are not well-defined as maps on S^1, since their changes of phase on $[0, 2\pi]$ equal $-(2d-1)\pi$ and $(2d-1)\pi$, respectively. Note that, in particular, $u_1^{(\varepsilon)}(2\pi) = u_2^{(\varepsilon)}(2\pi) = -1$. Therefore, we modify $u_1^{(\varepsilon)}, u_2^{(\varepsilon)}$ slightly to maps $\tilde{u}_j^{(\varepsilon)}, j = 1, 2$, in a similar manner to the argument used in the proof of Lemma 2.2. First, we use the rescaling (2.4) to define $\tilde{u}_j^{(\varepsilon)}, j = 1, 2$, on $[0, 2\pi - \varepsilon]$. Then, on $(2\pi - \varepsilon, 2\pi]$ we set

$$\tilde{u}_1^{(\varepsilon)}(\theta) = \tilde{u}_2^{(\varepsilon)}(\theta) = -\exp\Big(i\pi\frac{\theta - (2\pi - \varepsilon)}{\varepsilon}\Big).$$

Evidently, $\tilde{u}_2^{(\varepsilon)} \in \mathcal{E}_d, \tilde{u}_1^{(\varepsilon)} \in \mathcal{E}_{1-d}$ and a simple computation yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{S^1} |(u_2^{(\varepsilon)} - u_1^{(\varepsilon)})'|^2 &= \lim_{\varepsilon\to 0}\int_{S^1} ||u_2^{(\varepsilon)} - u_1^{(\varepsilon)}|'|^2 \\ &= \lim_{\varepsilon\to 0}\int_{S^1} |(\rho^{(\varepsilon)})'|^2 = \int_{S^1} (\rho')^2 = \frac{8(2d-1)^2}{\pi}.\end{aligned}$$

(ii) Assume by negation that there exist $u_1 \in \mathcal{E}_{d_1}$ and $u_2 \in \mathcal{E}_{d_2}$ such that

$$\int_{S^1} |(u_1 - u_2)'|^2 = \delta^2(d_1, d_2). \tag{2.10}$$

We may assume without loss of generality that $u_2(0) = u_1(0)$. It then follows from (2.2)–(2.3) that the function $|u_2 - u_1|$ must be equal to the function ρ given by (2.6). On the interval $K = [\frac{\pi}{2m}, \frac{3\pi}{2m}]$ we may write $u_2 - u_1 = \rho e^{i\phi}$, so that

$$\int_K |(u_2 - u_1)'|^2 = \int_K (\rho')^2 + \rho^2(\phi')^2.$$

Hence, using (2.10) and (2.3) we infer that ϕ is identically equal to a constant on K. Without loss of generality we may assume that the constant is equal to $\pi/2$. Therefore, $u_1 = \bar{u}_2$ on K, where we may write $u_2 = e^{i\psi}, u_1 = e^{-i\psi}$. It follows that $\rho = 2\sin\psi$, i.e., $\psi = \sin^{-1}(\rho/2)$ on K. Because of the nature of the singularity of ψ' at $\theta = \pi/m$, the function ψ does not belong to $H^1(K)$. Hence also $u_2 = e^{\psi} \notin H^1(K, S^1)$, contradicting our starting assumption. ∎

Remark 2.1: It is, of course, possible to consider the distance between homotopy classes with respect to other norms. For example, one may work in $W^{1,p}(S^1, S^1)$, $p \in [1, \infty]$, and consider the distance

$$\delta_{(p)}(d_1, d_2) = \inf\{\| (u_1 - u_2)'\|_{L^p(S^1)} : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}\}.$$

By the above techniques we get:

$$\delta_{(p)}(d_1, d_2) = \begin{cases} \frac{2^{1+1/p}m}{\pi^{1-1/p}} & 1 \le p < \infty, \\ \frac{2m}{\pi} & p = \infty. \end{cases}$$

However, the situation may be different when working with weaker norms. For example, although maps in $H^{1/2}(S^1, S^1)$ have a well-defined degree, it was shown by Brezis and Nirenberg [3, Lemma 6 and Remark 6] that for all d_1 and d_2,

$$\inf\{| u_1 - u_2|_{H^{1/2}} : u_1 \in \mathcal{E}_{d_1}, u_2 \in \mathcal{E}_{d_2}\} = 0,$$

where $| \cdot |_{H^{1/2}}$ denotes the $H^{1/2}$-seminorm.

It is also straightforward to extend the statements and arguments presented above to the case where the entire Sobolev norm is used as a distance function between homotopy classes.

3. S^1-valued maps on multiply connected domains in $\mathbb{R}^2$

Let $G, \omega_1, \ldots, \omega_n$ be smooth bounded simply connected domains in $\mathbb{R}^2$ with $\omega_j \subset\subset G$ for all j, and let $\Omega = G \setminus \bigcup_{j=1}^n \omega_j$ (see a scketch in Figure 1).

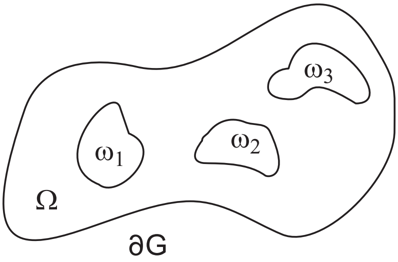

Figure 1. The domain Ω

The orientation with respect to which we shall define degrees in the sequel is: positive (i.e., counter-clockwise) on $\partial\omega_j, j = 1, \dots, n$ and clockwise on $\partial\omega_0 := \partial G$. Set

$$\mathcal{V}_0 = \Big\{ \boldsymbol{d} = (d_0, d_1, \dots, d_n) \in \mathbb{Z}^{n+1} : \sum_{j=0}^{n} d_j = 0 \Big\}. \tag{3.1}$$

For $\boldsymbol{d} \in \mathcal{V}_0$ define

$$\mathcal{E}_{\boldsymbol{d}} = \{ v \in H^1(\Omega, S^1) : \deg(v, \partial\omega_j) = d_j, j = 0, 1, \dots, n \}.$$

We therefore obtain a disjoint decomposition

$$H^1(\Omega, S^1) = \bigcup_{\boldsymbol{d} \in \mathcal{V}_0} \mathcal{E}_{\boldsymbol{d}}.$$

We shall investigate the distance $\delta(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)})$ as defined in (1.2) (for $D = \Omega$) for any $\boldsymbol{d}^{(1)} \neq \boldsymbol{d}^{(2)}$ in $\mathcal{V}_0$. We first recall some known results on the minimum of the energy in each homotopy class. Let

$$I(\boldsymbol{d}) = \inf_{v \in \mathcal{E}_{\boldsymbol{d}}} \int_\Omega |\nabla v|^2. \tag{3.2}$$

By [1, Th. I.1] we have

$$I(\boldsymbol{d}) = \int_\Omega |\nabla \Phi|^2,$$

where $\Phi = \Phi_{\boldsymbol{d}}$ satisfies, for some unprescribed constants $C_1, \dots, C_n$,

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega, \\ \Phi = C_j & \text{on } \partial\omega_j, j = 1, \dots, n, \\ \Phi = 0 & \text{on } \partial G, \\ \int_{\partial\omega_j} \frac{\partial\Phi}{\partial\nu} = 2\pi d_j, & j = 1, \dots, n. \end{cases}$$

Moreover, the infimum in (3.2) is attained by a unique u, up to a constant rotation, which satisfies

$$\begin{cases} u \times \frac{\partial u}{\partial x_1} = -\frac{\partial \Phi}{\partial x_2}, \\ u \times \frac{\partial u}{\partial x_2} = \frac{\partial \Phi}{\partial x_1}. \end{cases}$$

We also have $\Phi = \sum_{j=1}^n d_j \Phi_j$, where for each $j = 1, \dots, n$, Φ_j satisfies

$$\begin{cases} \Delta\Phi_j = 0 & \text{in } \Omega, \\ \Phi_j = \text{const} & \text{on } \partial\omega_j, j = 1, \dots, n, \\ \Phi_j = 0 & \text{on } \partial G, \\ \int_{\partial\omega_i} \frac{\partial\Phi_j}{\partial\nu} = 2\pi\delta_{i,j}, & i = 1, \dots, n. \end{cases}$$

Then,

$$I(\boldsymbol{d}) = \int_\Omega |\nabla\Phi|^2 = \sum_{i,j=1}^n c_{i,j} d_i d_j,$$

where $c_{i,j} := \int_\Omega \nabla\Phi_i \nabla\Phi_j$.

Next, for any $u_j \in \mathcal{E}_{\boldsymbol{d}^{(j)}}$, $j = 1, 2$, we have $v := u_2/u_1 \in \mathcal{E}_{\boldsymbol{d}}$, with $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)}$. Clearly

$$\int_\Omega |\nabla(u_2 - u_1)|^2 \geq \int_\Omega \big|\nabla|u_2 - u_1|\big|^2 = \int_\Omega \big|\nabla|v - 1|\big|^2. \tag{3.3}$$

Defining

$$H(\boldsymbol{d}) := \inf\left\{ \int_\Omega \big|\nabla|w - 1|\big|^2 : w \in \mathcal{E}_{\boldsymbol{d}} \right\}, \tag{3.4}$$

we therefore proved the following lower bound.

LEMMA 3.1: *For every $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \in \mathcal{V}_0$ we have $\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \geq H(\boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)})$.*

The next proposition provides an explicit relation between $H(\boldsymbol{d})$ and $I(\boldsymbol{d})$.

PROPOSITION 3.1: *For every $\boldsymbol{d} \in \mathcal{V}_0$ we have*

$$H(\boldsymbol{d}) = \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}). \tag{3.5}$$

Furthermore, the infimum in (3.4) *is never attained (for $\boldsymbol{d} \neq \boldsymbol{0}$).*

Proof: Consider the map $T\colon S^1 \to S^1$ defined as follows. For any $e^{i\phi} \in S^1$, $\phi \in (-\pi, \pi]$, let $T(e^{i\phi}) = e^{i\theta}$ with $\theta = \pi \sin\phi/2$. Note that

$$|e^{i\phi} - 1| = 2\left|\sin\frac{\phi}{2}\right| = \left(\frac{2}{\pi}\right)|\theta|. \tag{3.6}$$

Let the operator $\mathcal{T}\colon H^1(\Omega, S^1) \to H^1(\Omega, S^1)$ be defined for any $w \in H^1(\Omega, S^1)$ by $(\mathcal{T}w)(x) = T(w(x)), \forall x \in \Omega$. Since T is a bijective C^1-map from S^1 to S^1, $\mathcal{T}$ sends each $\mathcal{E}_{\boldsymbol{d}}$ to itself. By (3.6)

$$\int_\Omega \big|\nabla|w - 1|\big|^2 = \left(\frac{2}{\pi}\right)^2 \int_\Omega |\nabla(\mathcal{T}w)|^2, \quad \forall w \in \mathcal{E}_{\boldsymbol{d}}, \tag{3.7}$$

and it follows that $H(\boldsymbol{d}) \geq (\frac{2}{\pi})^2 I(\boldsymbol{d})$.

Next we turn to the proof of the reverse inequality. The inverse $S = T^{-1}$ of T is given by: $S(e^{i\theta}) = e^{i\phi}$, with $\phi = 2\sin^{-1}\theta/\pi, \forall\theta \in (-\pi, \pi]$. This map is continuous but not Lipschitz. We therefore define, for each small $\varepsilon > 0$, an approximation S_ε by:

$$S_\varepsilon(e^{i\theta}) = e^{i\phi} \text{ with } \phi = 2\sin^{-1}\left(J_\varepsilon\left(\frac{\theta}{\pi}\right)\right), \quad \forall\theta \in (-\pi, \pi], \tag{3.8}$$

where J_ε is defined in (2.7). Since $|S_\varepsilon(e^{i\theta}) - 1| = 2|J_\varepsilon(\theta/\pi)|$ it follows from (2.7) that

$$\left|\frac{d}{d\theta}(|S_\varepsilon(e^{i\theta}) - 1|)\right| \le C, \quad \forall\theta, \forall\varepsilon.$$

Therefore, defining for each $W \in \mathcal{E}_{\boldsymbol{d}}$,

$$(\mathcal{S}_\varepsilon W)(x) = S_\varepsilon(W(x)), \quad \forall x \in \Omega, \tag{3.9}$$

we have $\mathcal{S}_\varepsilon W \in \mathcal{E}_{\boldsymbol{d}}$ and

$$\lim_{\varepsilon\to 0}\int_\Omega \big|\nabla|\mathcal{S}_\varepsilon W - 1|\big|^2 = \int_\Omega \big|\nabla|S(W) - 1|\big|^2. \tag{3.10}$$

From (3.10) and (3.6) we finally infer that

$$\lim_{\varepsilon\to 0}\int_\Omega \big|\nabla|\mathcal{S}_\varepsilon W - 1|\big|^2 = \left(\frac{2}{\pi}\right)^2 \int_\Omega |\nabla W|^2, \tag{3.11}$$

which yields $H(\boldsymbol{d}) \le (\frac{2}{\pi})^2 I(\boldsymbol{d})$.

Finally, we show that the infimum in (3.4) is not attained. Looking for a contradiction, assume that it is attained by some $w \in \mathcal{E}_{\boldsymbol{d}}$. From (3.5) and (3.7) it then follows that $W := \mathcal{T}w$ must be a minimizer in (3.2). We recall that by a result of [1] (see the beginning of this section), W is a C^∞ map that can be written locally in Ω as $W = e^{i\psi}$ with ψ a smooth harmonic function. It is a standard fact that the critical points of a nonconstant harmonic function are isolated and that the level-set through a critical point z_0 consists locally of two or more analytic curves intersecting at z_0 with equal angles (cf. [4, pages 18–19]). Therefore, regardless of the property of -1 being a critical or a regular value of W, there exists a subsegment of an analytic curve, $S \subset \{W = -1\}$, on which $\psi \equiv \pi$ and $|\nabla W| = |\nabla\psi| \ne 0$. We can then choose a narrow enough "tube", $D = \{x \in \Omega : \text{dist}(x, S) < \varepsilon\}$, such that $|\nabla\psi| \ge \eta > 0$ on D. But then it follows easily that the function $\phi = 2\sin^{-1}\psi/\pi$ does not belong to $H^1(D)$. Therefore, $w = e^{i\phi} \notin H^1(D, S^1)$, which is a contradiction. ∎

The next result is a direct consequence of Lemma 3.1 and Proposition 3.1.

THEOREM 2: *For every* $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \in \mathcal{V}_0$ *we have*

$$\delta^2(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}}) \geq \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}^{(1)} - \boldsymbol{d}^{(2)}). \tag{3.12}$$

Another consequence is a (partial) analogue to assertion (ii) of Theorem 1 (see open problem 1 at the end of this section).

COROLLARY 3.1: *If* $\boldsymbol{d}^{(1)} \neq \boldsymbol{d}^{(2)}$ *are such that equality holds in* (3.12) *then* $\delta(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}})$ *is not attained.*

Proof: Assume by contradiction, that $\delta(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}})$ is attained by the two maps $u_j \in \mathcal{E}_{\boldsymbol{d}^{(j)}}, j = 1, 2$. Then, it follows from (3.3) that $v = u_2/u_1 \in \mathcal{E}_{\boldsymbol{d}}$, with $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)}$, realizes the infimum in (3.4), and thus contradicting Proposition 3.1. ∎

Next we look for conditions that guarantee that the inequality in (3.12) is actually an equality. The following theorem shows that this is the case when $\boldsymbol{d}^{(2)} = -\boldsymbol{d}^{(1)}$.

THEOREM 3: *If* $\boldsymbol{d}^{(2)} = -\boldsymbol{d}^{(1)}$ *then the lower bound of Theorem 2 is sharp.*

Proof: Put $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)} = 2\boldsymbol{d}^{(2)}$. By Proposition 3.1 and the density of $C^1(\Omega, S^1)$ in $H^1(\Omega, S^1)$ (see [2]), for every $\varepsilon > 0$ there exists $w_\varepsilon \in C^1(\Omega, S^1) \cap \mathcal{E}_{\boldsymbol{d}}$ such that

$$\int_\Omega \left|\nabla |w_\varepsilon - 1|\right|^2 \leq \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}) + \varepsilon.$$

Since d_j is even for all j it follows that there exists $u_\varepsilon \in \mathcal{E}_{\boldsymbol{d}^{(2)}} \cap C^1(\Omega, S^1)$ such that $u_\varepsilon^2 = w_\varepsilon$. Put $v_\varepsilon = 1/u_\varepsilon = \bar{u}_\varepsilon$ which belongs to $\mathcal{E}_{\boldsymbol{d}^{(1)}} \cap C^1(\Omega, S^1)$. Clearly, $|u_\varepsilon - v_\varepsilon| = |w_\varepsilon - 1|$ and $\Re(u_\varepsilon - v_\varepsilon) = \Re(u_\varepsilon - \bar{u}_\varepsilon) = 0$. Therefore,

$$\int_\Omega |\nabla(u_\varepsilon - v_\varepsilon)|^2 = \int_\Omega \left|\nabla |u_\varepsilon - v_\varepsilon|\right|^2 = \int_\Omega \left|\nabla |w_\varepsilon - 1|\right|^2 \leq \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}) + \varepsilon.$$

The result follows since ε is arbitrary. ∎

A second case where equality holds in (3.12) is given by the next theorem. In order to describe it we introduce a property of certain vectors $\boldsymbol{d} \in \mathcal{V}_0$. For such $\boldsymbol{d}$ we fix a minimizer W for $I(\boldsymbol{d})$ over $\mathcal{E}_{\boldsymbol{d}}$ (see (3.2)). Recall that all the minimizers in (3.2) are given by $\{e^{i\alpha}W : \alpha \in (-\pi, \pi]\}$. Note that by Sard's theorem, for a.e. $\alpha \in (-\pi, \pi]$, $e^{i\alpha}$ is a regular value of W. By this we mean that $e^{i\alpha}$ is a regular value of both $W|_\Omega$ and of $W|_{\partial\Omega}$. Therefore, denoting $g := W|_{\partial\Omega}$, we have: $g^{-1}(e^{i\alpha})$ consists of a finite number of points $x_1, \ldots, x_m \in \partial\Omega$ and

$W^{-1}(e^{i\alpha})$ is a union of smooth curves, each connecting some x_i to an x_j. In fact, $W^{-1}(e^{i\alpha})\cap\Omega$ cannot include closed loops since this would violate the minimizing property of W (we would redefine $W \equiv e^{i\alpha}$ inside the loop, hence decreasing its energy). For each such α consider a graph $\mathcal{G}_\alpha$ with vertices $y_0, y_1, \dots, y_n$ such that y_j corresponds to $\partial\omega_j$, for $j = 0, 1, \dots, n$. For $i \neq j$ there is an edge in $\mathcal{G}_\alpha$ between y_i and y_j if and only if $W^{-1}(e^{i\alpha})$ contains a curve joining two points, one in $\partial\omega_i$ and the other one in $\partial\omega_j$.

Definition 3.1: We shall say that $\boldsymbol{d}$ has **property** (C) if for a minimizer W in (3.2) there exists $\alpha \in (-\pi, \pi]$ for which the graph $\mathcal{G}_\alpha$ is connected.

THEOREM 4: *Assume that $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \in \mathcal{V}_0$ are such that $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)}$ has property (C). Then,*

$$\delta^2(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}}) = \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}). \tag{3.13}$$

Proof: We may assume without loss of generality that property (C) is satisfied for W with $\alpha = 0$. Let $g := W|_{\partial\Omega}$ and $g^{-1}(1) = \{x_1, \dots, x_m\} \subset \partial\Omega$. To each x_j we associate a sign $s_j \in \{\pm 1\}$ as follows. Let $x_j \in \partial\omega_k$ for some $k \in \{0, 1, \dots, n\}$. In a boundary interval around x_j in $\partial\omega_k$ we may write $g = e^{i\phi}$. We set $s_j = 1$ if ϕ is increasing with respect to the orientation that we fixed above on $\partial\Omega$ and $s_j = -1$ otherwise. Put

$$P^+ = \{x_j : s_j = 1\} \quad \text{and} \quad P^- = \{x_j : s_j = -1\}.$$

It is easy to see that $|P^+| = |P^-| = m/2$ (i.e., m is even), and then $W^{-1}(1)$ consists of $m/2$ disjoint curves, each connecting a point from P^+ to a point from P^-. By the definition of the degree we have

$$\sum_{x_j \in \partial\omega_i} s_j = d_i, \quad i = 0, 1, \dots, n.$$

By assumption (C), $\mathcal{G}_0$ is connected, so there exists a spanning tree with n edges, $e_1, \dots, e_n$. To each edge e_j we associate two indices $i_+(j), i_-(j)$ such that e_j connects $y_{i_+(j)}$ (corresponding to a positive point on $\partial\omega_{i_+(j)}$) to $y_{i_-(j)}$. To e_j corresponds a curve $\gamma_j \subset W^{-1}(1)$ connecting $\partial\omega_{i_+(j)}$ to $\partial\omega_{i_-(j)}$.

For any small $\varepsilon > 0$ define a map $w_\varepsilon \in \mathcal{E}_{\boldsymbol{d}}$ by $w_\varepsilon = \mathcal{S}_\varepsilon W$ (see (3.8)–(3.9)). Note that by definition $w_\varepsilon^{-1}(1) = W^{-1}(1)$. Moreover, by (3.11)

$$\lim_{\varepsilon \to 0} \int_\Omega \left|\nabla |w_\varepsilon - 1|\right|^2 = \left(\frac{2}{\pi}\right)^2 \int_\Omega |\nabla W|^2 = \left(\frac{2}{\pi}\right)^2 I(\boldsymbol{d}). \tag{3.14}$$

Fix some $\varepsilon > 0$ and for small $\eta > 0$ consider a "tube" of width η around each γ_j:

$$T_\eta^{(j)} = \{x \in \overline{\Omega} : \mathrm{dist}(x, \gamma_j) < \eta\}.$$

One can readily check that there exists a C^1 map $z_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{d}}$ with the following properties:

(i) $z_{\varepsilon,\eta} \equiv 1$ on $T_{\eta/2}^{(j)}$, $j = 1, \dots, n$.
(ii) $z_{\varepsilon,\eta} = w_\varepsilon$ on $A_\eta := \Omega \setminus \bigcup_{j=1}^n T_\eta^{(j)}$.
(iii) $\int_\Omega |\nabla(z_{\varepsilon,\eta} - w_\varepsilon)|^2 = O_\varepsilon(\eta)$.

Since $A_{\eta/2}$ is simply connected there exists $u_{\varepsilon,\eta} \in C^1(A_{\eta/2}, S^1)$ such that $u_{\varepsilon,\eta}^2 = z_{\varepsilon,\eta}$ in $A_{\eta/2}$. Put $v_{\varepsilon,\eta} = \bar{u}_{\varepsilon,\eta} = 1/u_{\varepsilon,\eta}$.

For each $j \in \{1, \dots, n\}$ consider the two "long sides" of $\partial T_{\eta/2}^{(j)}$, γ_j^- and γ_j^+, so that the ordering $\{\gamma_j^-, \gamma_j, \gamma_j^+\}$ corresponds to the orientation that we defined on $\partial\omega_{i_+(j)}$. We have $u_{\varepsilon,\eta} = v_{\varepsilon,\eta} \equiv \sigma_j^-$ on γ_j^- and $u_{\varepsilon,\eta} = v_{\varepsilon,\eta} \equiv \sigma_j^+$ on γ_j^+ with $\sigma_j^\pm \in \{-1, 1\}$. Put $\sigma_j = \sigma_j^+/\sigma_j^-$. Take any smooth function h_j on $\overline{T}_{\eta/2}^{(j)}$ that satisfies

$$h_j \equiv 0 \text{ on } \gamma_j^- \quad \text{and} \quad h_j \equiv \pi\Big(\frac{1-\sigma_j}{2}\Big) \text{ on } \gamma_j^+.$$

Then, complete the definition of $u_{\varepsilon,\eta}$ and $v_{\varepsilon,\eta}$ to $\bigcup_{j=1}^n T_{\eta/2}^{(j)}$ by setting

$$u_{\varepsilon,\eta} = v_{\varepsilon,\eta} = \sigma_j^- e^{ih_j} \text{ on } T_{\eta/2}^{(j)}, \quad j = 1, \dots, n.$$

It follows that for some $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)} \in \mathcal{V}_0$ we have $u_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{e}^{(1)}}$ and $v_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{e}^{(2)}}$. Since $u_{\varepsilon,\eta}/v_{\varepsilon,\eta} = z_{\varepsilon,\eta}$ in Ω, we still have

$$\boldsymbol{e}^{(2)} - \boldsymbol{e}^{(1)} = \boldsymbol{d}. \tag{3.15}$$

For each $j = 1, \dots, n$ define a vector $\boldsymbol{a}^{(j)} \in \mathcal{V}_0$ by:

$$a_i^{(j)} = \begin{cases} 1 & \text{if } i = i_+(j), \\ -1 & \text{if } i = i_-(j), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that each vector in $\mathcal{V}_0$ can be expressed as a linear combination, with integer coefficients, of $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}$. In particular, for $\boldsymbol{b} := \boldsymbol{d}^{(1)} - \boldsymbol{e}^{(1)} = \boldsymbol{d}^{(2)} - \boldsymbol{e}^{(2)}$ (see (3.15)) there exist $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ such that

$$\boldsymbol{b} = \sum_{j=1}^n \alpha_j \boldsymbol{a}^{(j)}.$$

For each $j = 1, \dots, n$, take a smooth function H_j on $\overline{T}_{\eta/2}^{(j)}$ that satisfies

$$H_j \equiv 0 \text{ on } \gamma_j^- \quad \text{and} \quad H_j \equiv 2\pi\alpha_{i_+(j)} \text{ on } \gamma_j^+.$$

Then define a function H on Ω by

$$H(x) = \begin{cases} H_j(x) & \text{if } x \in T^{(j)}_{\eta/2} \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, set

$$\tilde{u}_{\varepsilon,\eta} = e^{iH} u_{\varepsilon,\eta} \quad \text{and} \quad \tilde{v}_{\varepsilon,\eta} = e^{iH} v_{\varepsilon,\eta}.$$

We have that $\tilde{u}_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{d}^{(1)}}, \tilde{v}_{\varepsilon,\eta} \in \mathcal{E}_{\boldsymbol{d}^{(2)}}$ and $\tilde{u}_{\varepsilon,\eta}/\tilde{v}_{\varepsilon,\eta} = z_{\varepsilon,\eta}$. Therefore,

$$|\tilde{u}_{\varepsilon,\eta} - \tilde{v}_{\varepsilon,\eta}| = |z_{\varepsilon,\eta} - 1| \quad \text{in } \Omega.$$

Combining it with property (iii) of $z_{\varepsilon,\eta}$ and (3.14) we get that

$$\begin{aligned}\lim_{\eta\to 0} \int_\Omega |\nabla(\tilde{u}_{\varepsilon,\eta} - \tilde{v}_{\varepsilon,\eta})|^2 = \lim_{\eta\to 0} \int_\Omega \big|\nabla|\tilde{u}_{\varepsilon,\eta} - \tilde{v}_{\varepsilon,\eta}|\big|^2 &= \lim_{\eta\to 0} \int_\Omega \big|\nabla|z_{\varepsilon,\eta} - 1|\big|^2 \\ &= \int_\Omega \big|\nabla|w_\varepsilon - 1|\big|^2 + o_\varepsilon(1) \\ &= \Big(\frac{2}{\pi}\Big)^2 I(\boldsymbol{d}) + o_\varepsilon(1).\end{aligned}$$

Letting $\varepsilon \to 0$ we infer that $\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \leq (\frac{2}{\pi})^2 I(\boldsymbol{d})$ which together with Theorem 2 gives the result (3.13). ∎

Remark 3.1: Property (C) is satisfied, for example, when $d_1, d_2, \ldots, d_n > 0$. More generally, it is satisfied if there is no nontrivial subset of indices $J \subsetneq \{0, 1, \ldots, n\}$ for which $\sum_{i\in J} d_i = 0$. On the other hand, we shall give an example in the next section that shows that both property (C) and the conclusion of Theorem 4 do not always hold.

We close this section with two open problems:

Open problem 1: *Is it true that $\delta(\mathcal{E}_{\boldsymbol{d}^{(1)}}, \mathcal{E}_{\boldsymbol{d}^{(2)}})$ is never attained for $\boldsymbol{d}^{(1)} \neq \boldsymbol{d}^{(2)}$, i.e., is the assumption made in Corollary 3.1, that equality holds in (3.12), unnecessary?*

Open problem 2: *Given $\boldsymbol{d} \neq \boldsymbol{0}$, is property (C) a necessary condition for the equality in (3.13) to hold for all $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}$ satisfying $\boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)} = \boldsymbol{d}$?*

4. An example

In the beginning of this section we shall study the distance between homotopy classes of S^1-valued maps defined on a certain **graph**. We shall later use it to produce an example for which the conclusion of Theorem 4 does not hold. Consider the graph $G = F \cup E_1 \cup E_2$, where $F = \partial B_1(0)$, $E_1 = \partial B_1(2e^{is})$ and $E_2 = \partial B_1(2e^{-is})$, for some $s \in (\pi/6, \pi/2)$, so that E_1 and E_2 do not intersect each other and they touch F at the points e^{is} and e^{-is}, respectively. Here $B_1(p)$ is the unit disc centered at p. We define:

$$H^1(G, S^1) := C(G, S^1) \cap H^1(F, S^1) \cap H^1(E_1, S^1) \cap H^1(E_2, S^1).$$

To each $u \in H^1(G, S^1)$ we associate the vector $\boldsymbol{d} = (d_1, d_2, d_3)$ of the degrees of u on F, E_1 and E_2, respectively. This induces a decomposition $H^1(G, S^1) = \bigcup_{\boldsymbol{d} \in \mathbb{Z}^3} \mathcal{E}_{\boldsymbol{d}}$ as in the previous sections. The distance between two homotopy classes is given by, analogously to (1.2) and (2.1),

$$\delta_G^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) = \inf \left\{ \int_G |(u_1 - u_2)'|^2 : u_1 \in \mathcal{E}_{\boldsymbol{d}^{(1)}}, u_2 \in \mathcal{E}_{\boldsymbol{d}^{(2)}} \right\}. \tag{4.1}$$

We shall need the following simple lemma whose proof is postponed after the description of our example.

LEMMA 4.1: *Let $u, v \in H^1(S^1, S^1)$ satisfy $\deg u = \deg v = k \neq 0$ and $|(u - v)(1)| = \eta > 0$. Then,*

$$\int_{S^1} |(u - v)'|^2 \geq \frac{2\eta^2}{\pi}, \tag{4.2}$$

and this bound is optimal.

The next result gives an example of non-optimality of the bound of Theorem 2 for maps in $H^1(G, S^1)$.

PROPOSITION 4.1: *Let $\boldsymbol{d}^{(1)} = (1, 1, 1)$ and $\boldsymbol{d}^{(2)} = (2, 1, 1)$. Then,*

$$\delta_G^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) > \frac{8}{\pi}. \tag{4.3}$$

Proof: Take any $u_1 \in \mathcal{E}_{\boldsymbol{d}^{(1)}}$ and $u_2 \in \mathcal{E}_{\boldsymbol{d}^{(2)}}$ and let $w = u_2 - u_1$. Put $\eta = \max(|w(e^{is})|, |w(e^{-is})|)$ which we may assume without loss of generality to be achieved at $e^{is} = E_1 \cap F$. By Lemma 4.1

$$\int_{E_1} |w'|^2 \geq \frac{2\eta^2}{\pi}. \tag{4.4}$$

Since at some point on F we must have $|w| = 2$, we deduce from Theorem 1(i) and the argument used there, that for some $s_1, s_2 \geq 0$ satisfying $s_1 + s_2 = 2s$ or $s_1 + s_2 = 2\pi - 2s$ we have

$$\int_F |w'|^2 \geq \max\Big(\frac{8}{\pi}, \frac{(2-\eta)^2}{s_1} + \frac{(2-\eta)^2}{s_2}\Big). \tag{4.5}$$

From (4.4)–(4.5) we infer

$$\int_G |w'|^2 \geq g(\eta) := \frac{2\eta^2}{\pi} + \max\Big(\frac{8}{\pi}, \frac{4(2-\eta)^2}{2\pi - 2s}\Big). \tag{4.6}$$

Evidently,

$$\gamma_0 := \min_{\eta \in [0,2]} g(\eta) > \frac{8}{\pi}, \tag{4.7}$$

and the result follows. ∎

Proof of Lemma 4.1: Put $w = u - v$. We may assume without loss of generality that $w(1) = (u-v)(1) = \eta i$. There are two possibilities:

(i) $w(e^{i\theta_0}) = 0$ for some $\theta_0 \in (0, 2\pi)$. Then, by the Cauchy–Schwarz inequality

$$\int_{S^1} |w'|^2 \geq \int_{S^1} \big||w|'\big|^2 \geq \frac{\eta^2}{\theta_0} + \frac{\eta^2}{2\pi - \theta_0} \geq \frac{2\eta^2}{\pi}.$$

(ii) $\gamma := \min\{|w(e^{i\theta})| : \theta \in [0, 2\pi)\} > 0$. In this case the winding number of w with respect to 0 is also k. In particular, there is some $\theta_1 \in (0, 2\pi)$ for which $w(e^{i\theta_1}) = -ti$, with $t \geq \gamma$. Writing $w = w_1 + iw_2$ we have

$$\int_{S^1} |w'|^2 \geq \int_{S^1} |w_2'|^2 \geq \frac{(\eta+t)^2}{\theta_1} + \frac{(\eta+t)^2}{2\pi - \theta_1} \geq 2\frac{(\eta+t)^2}{\pi} > \frac{2\eta^2}{\pi},$$

and (4.2) follows in this case too.

Finally, we show that (4.2) is optimal. We use a construction similar to the one used in the proof of Theorem 1(i). Assume first that $\eta < 2$. Fix any small $\varepsilon > 0$. First, set

$$u^{(\varepsilon)}(\theta) = v^{(\varepsilon)}(\theta) = \exp\Big(2\pi k \frac{\theta + \pi}{\varepsilon} i\Big), \quad \theta \in [-\pi, \varepsilon - \pi].$$

On $[\varepsilon - \pi, 0]$ let $\rho(\theta) = \rho^{(\varepsilon)}(\theta)$ be the linear function satisfying $\rho(\varepsilon - \pi) = 0$ and $\rho(0) = \eta$. Let $\alpha(\theta) = \sin^{-1} \rho(\theta)/2$ and then set

$$u^{(\varepsilon)}(\theta) = e^{i\alpha(\theta)} \ \text{ and } \ v^{(\varepsilon)}(\theta) = e^{-i\alpha(\theta)}, \quad \theta \in [\varepsilon - \pi, 0].$$

Finally, on $[0,\pi]$ use a similar construction to the one used on $[\varepsilon-\pi,0]$, to get $u^{(\varepsilon)}$ and $v^{(\varepsilon)}$ conjugate to each other, and $|u^{(\varepsilon)}-v^{(\varepsilon)}|$ equals to a linear function changing from η back to 0. It is elementary to check that the above construction yields

$$\int_{S^1} |(u^{(\varepsilon)}-v^{(\varepsilon)})'|^2 = \eta^2\Big(\frac{1}{\pi-\varepsilon}+\frac{1}{\pi}\Big).$$

The case $\eta=2$ follows from the case $\eta<2$ by a standard approximation argument. ∎

Next we use the example of Proposition 4.1 to construct a perforated domain in $\mathbb{R}^2$ for which a case of strict inequality in (3.12) occurs. This domain is obtained by a certain thickening of the graph G. For any small ε consider an annulus $F^{(\varepsilon)}=B_{1+\varepsilon}(0)\backslash B_{1-\varepsilon}(0)$ and two other annuli $E_1^{(\varepsilon)}=B_{1+\varepsilon}(z_1)\backslash B_{1-\varepsilon}(z_1)$, $E_2^{(\varepsilon)}=B_{1+\varepsilon}(z_2)\setminus B_{1-\varepsilon}(z_2)$ with $z_1=(2+3\varepsilon)e^{is}$ and $z_2=(2+3\varepsilon)e^{-is}$. Finally, connect $F^{(\varepsilon)}$ to $E_1^{(\varepsilon)}$ and $E_2^{(\varepsilon)}$ by adding two small "tubes", $Q_1^{(\varepsilon)}$ and $Q_2^{(\varepsilon)}$ of length and width $\sim\varepsilon$, around the lines of phase $\theta=\pm s$. The resulting domain will be denoted by $G^{(\varepsilon)}$ (see Figure 2). We would like to extend the arguments of Proposition 4.1 to the new domain. Note that strictly speaking the degree vectors now are four-dimensional (see (3.1)), with d_0 corresponding to the degree along the outer boundary. However, with a slight abuse of notation we shall suppress d_0 and stick to the notation of the one-dimensional case.

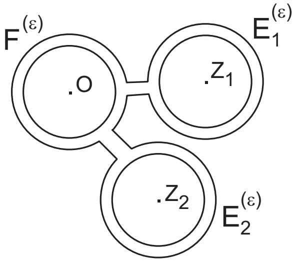

Figure 2. The domain $G^{(\varepsilon)}$

We have the following two-dimensional analogue to Proposition 4.1.

PROPOSITION 4.2: *Let $\boldsymbol{d}^{(1)}$ and $\boldsymbol{d}^{(2)}$ be as in Proposition 4.1. Then, for small enough ε we have*

$$\delta^2(\boldsymbol{d}^{(1)},\boldsymbol{d}^{(2)}) > H(\boldsymbol{d}^{(2)}-\boldsymbol{d}^{(1)}). \tag{4.8}$$

Proof: Let $\boldsymbol{d} = \boldsymbol{d}^{(2)} - \boldsymbol{d}^{(1)} = (1, 0, 0)$. The inequality (4.8) is a direct consequence of (4.3) and the following two estimates:

$$H(\boldsymbol{d}) = \frac{16\varepsilon}{\pi} + O(\varepsilon^2),$$

and

$$\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \geq 2\gamma_0\varepsilon + O(\varepsilon^{3/2}),$$

where γ_0 is defined in (4.7).

Proof of (4.9): For any $w \in \mathcal{E}_{\boldsymbol{d}}$ we have clearly

$$\int_{G^{(\varepsilon)}} |\nabla w|^2 \geq \int_{F^{(\varepsilon)}} |\nabla w|^2 \geq \int_{1-\varepsilon}^{1+\varepsilon} \int_0^{2\pi} |\frac{\partial w}{\partial \theta}|^2 \frac{d\theta}{r} dr \geq 2\pi \log\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big). \tag{4.11}$$

On the other hand, consider the map $w^{(\varepsilon)}$ defined by $w^{(\varepsilon)}(z) = z/|z|$ on F_ε, $w^{(\varepsilon)}(z) \equiv e^{is}$ on $E_1^{(\varepsilon)}$ and $w^{(\varepsilon)}(z) \equiv e^{-is}$ on $E_2^{(\varepsilon)}$. By a direct construction we can complete the definition of $w^{(\varepsilon)}$ to $Q_1^{(\varepsilon)} \cup Q_2^{(\varepsilon)}$ with an additional cost of energy $O(\varepsilon^2)$. Thus we get

$$\int_{G^{(\varepsilon)}} |\nabla w^{(\varepsilon)}|^2 = 2\pi \log\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big) + O(\varepsilon^2). \tag{4.12}$$

From (4.11)–(4.12) it follows that $I(\boldsymbol{d}) = 4\pi\varepsilon + O(\varepsilon^2)$ which together with (3.5) implies (4.9).

Proof of (4.10): An explicit simple construction shows that

$$\delta^2(\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) \leq C\varepsilon, \tag{4.13}$$

(here and in the sequel we denote by C different constants which do not depend on ε). Thanks to (4.13), it suffices to consider only pairs of maps $u_j \in \mathcal{E}_{\boldsymbol{d}^{(j)}}$, $j = 1, 2$, satisfying

$$\int_{G^{(\varepsilon)}} |\nabla(u_2 - u_1)|^2 \leq C\varepsilon.$$

Consider such a pair and set $w = u_2 - u_1$. For each $\alpha \in (s - \varepsilon/2, s + \varepsilon/2)$ consider the two segments:

$$l_1(\alpha) = \{z_1 + re^{i(\pi+\alpha)} : r > 0\} \cap (E_1^{(\varepsilon)} \cup F^{(\varepsilon)}),$$
$$l_2(\alpha) = \{z_2 + re^{i(\pi-\alpha)} : r > 0\} \cap (E_2^{(\varepsilon)} \cup F^{(\varepsilon)}).$$

By Fubini's theorem there exists $\alpha_0 = \alpha_0(\varepsilon) \in (s - \varepsilon/2, s + \varepsilon/2)$ such that

$$\int_{l_1(\alpha_0) \cup l_2(\alpha_0)} |\nabla w|^2 \leq C. \tag{4.14}$$

Note that (4.14) implies

$$(4.15)\quad |w(x_2)-w(x_1)| \le C|x_2-x_1|^{1/2} \le C\varepsilon^{1/2}, \quad \forall x_1,x_2\in l_j(\alpha_0),\ j=1,2.$$

For each $r\in(1-\varepsilon,1+\varepsilon)$ define the following four points:

$$p_j(r)=l_j(\alpha_0)\cap\partial B_r(z_j), \quad q_j(r)=l_j(\alpha_0)\cap\partial B_r(0), \quad j=1,2.$$

Put

$$\eta(r)=\max(|w(p_1(r))|,|w(p_2(r))|) \quad\text{and}\quad \zeta(r)=\max(|w(q_1(r))|,|w(q_2(r))|).$$

By the argument of Proposition 4.1 we have for each $r\in(1-\varepsilon,1+\varepsilon)$:

$$(4.16)\qquad \int_{\partial B_r(0)\cup\partial B_r(z_1)\cup\partial B_r(z_2)} |w'|^2 \ge \frac{2\eta^2(r)}{\pi r}+\frac{1}{r}\max\Big(\frac{8}{\pi},\frac{4(2-\zeta(r))^2}{2\pi-2s}\Big).$$

By (4.15) there exists a constant η_0 such that:

$$|\eta(r)-\eta_0|,|\zeta(r)-\eta_0|\le C\varepsilon^{1/2}, \quad \forall r\in(1-\varepsilon,1+\varepsilon).$$

Therefore, from (4.16) and (4.6)–(4.7) we infer that

$$(4.17)\qquad \int_{\partial B_r(0)\cup\partial B_r(z_1)\cup\partial B_r(z_2)} |w'|^2 \ge \frac{\gamma_0}{r}-C\varepsilon^{1/2}, \quad \forall r\in(1-\varepsilon,1+\varepsilon).$$

Integrating (4.17) over $r\in(1-\varepsilon,1+\varepsilon)$ yields (4.10). ∎

Remark 4.1: Let $\boldsymbol{e}^{(1)}=(1,0,0)$ and $\boldsymbol{e}^{(2)}=(2,0,0)$, so that

$$\boldsymbol{e}^{(2)}-\boldsymbol{e}^{(1)}=\boldsymbol{d}^{(2)}-\boldsymbol{d}^{(1)}=(1,0,0).$$

The argument of Theorem 1 shows that $\delta_G^2(\boldsymbol{e}^{(1)},\boldsymbol{e}^{(2)})=8/\pi$. Therefore, Proposition 4.1 shows not only that the bound of Theorem 2 is not optimal in $H^1(G,S^1)$, but also that $\delta_G(\boldsymbol{d}^{(1)},\boldsymbol{d}^{(2)})$ is not always a function of $\boldsymbol{d}^{(2)}-\boldsymbol{d}^{(1)}$ only. The same conclusion holds for $H^1(G^{(\varepsilon)},S^1)$. Indeed, we claim that $\delta^2(\boldsymbol{e}^{(1)},\boldsymbol{e}^{(2)})=H(\boldsymbol{e}^{(2)}-\boldsymbol{e}^{(1)})$. In this special case, we do not need property (C) in its full strength, and we can use (part of) the argument of Theorem 4 once we prove that there is a regular value $e^{i\alpha}$ ($\alpha\in[0,2\pi)$) of W, for which $W^{-1}(e^{i\alpha})$ contains a curve joining $\partial B_{1-\varepsilon}(0)$ to $\partial B_{1+\varepsilon}(0)\cap\partial G^{(\varepsilon)}$ (W denotes as usual a minimizer in (3.2)). A quick way to see that, is via the co-area formula. Denoting by $\mathcal{H}^1$ the one dimensional Hausdorff measure, we have

$$\begin{aligned}\int_0^{2\pi}\mathcal{H}^1(\{W^{-1}(e^{i\alpha})\cap E_1^{(\varepsilon)}\})d\alpha &= \int_{E_1^{(\varepsilon)}}|\nabla W| \\ &\le \Big(\int_{E_1^{(\varepsilon)}}|\nabla W|^2\Big)^{1/2}\cdot|E_1^{(\varepsilon)}|^{1/2} \\ &\le C\varepsilon.\end{aligned}$$

Therefore, for almost every $\alpha \in (0, 2\pi)$, $e^{i\alpha}$ is a regular value of W such that $W^{-1}(e^{i\alpha}) \cap E_1^{(\varepsilon)}$ is a curve of length $O(\varepsilon)$. The existence of an α with the desired properties follows.

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